CMB Anisotropies

Episode II: Attack of the $C_l$ ones

Approximation Methods &
Cosmological Parameter Dependencies

By Andy Friedman
Astronomy 200, Harvard University, Spring 2003
Outline

- **Elucidating the Physics of the Acoustic Peaks**
  - Two-Fluid Approximation (Seljak 1994)

- **Determining the Cosmological Parameters From the Temperature Power Spectrum**
  - Dependencies
  - Degeneracies

- **Parameter Visualizations**
  - Uros Seljak’s Plots (Two-Fluid Approx)
  - Max Tegmark’s Movies (CMBFAST)
  - Wayne Hu’s Movies (CMBFAST)
The positions and amplitudes of the acoustic peaks in the CMB power spectrum contain enough information to allow simultaneous determination of the cosmological Parameters, of which we are mainly interested in:

\[ H_0, \Omega_b, \Omega_m, \Omega_k, \text{ and } \Omega_\Lambda \text{ or equivalently,} \]

\[ h, \Omega_b h^2, \Omega_m h^2, \Omega_k h^2, \text{ and } \Omega_\Lambda h^2 \]

We hope to elucidate the main physical processes behind the acoustic peaks and isolate their effects on the most interesting cosmological parameters
Caveats

- We’ll restrict analysis to the temperature power spectrum (no polarization spectra).

- We’ll ignore vector and tensor contributions to the temperature power spectrum (i.e. vorticity & primordial gravity waves from inflation).

- We won’t discuss how the E and B mode polarization spectra can break cosmological parameter degeneracies.
Physics of Acoustic Oscillations

- The early universe is an acoustic cavity where the photon-baryon fluid can oscillate due to competing restoring forces of radiation pressure from the photons vs. gravitational compression from the matter.

- Since you get “hot spots” from compression and “cold spots” from rarefaction, there will be a harmonic series of peaks associated with the acoustic oscillations.

- The scale of the first of these peaks is determined by the size of the horizon at the decoupling (i.e. $x_\circ$ the angular distance to the last scattering surface (LSS)).

See Wayne Hu’s tutorial [http://background.uchicago.edu/~whu/physics/tour.html](http://background.uchicago.edu/~whu/physics/tour.html)
Critical Angles cont...

| \( \theta_1 \) \( \sim \) 1.8 \((\Omega)^{1/2}\) degrees | Set by ratio of co-moving horizon size \( D_{LS} \) at LSS to \( R_H \), the horizon size at present epoch: \( R_H = \frac{2c}{\Omega \Omega_m H_0} \) (open)  
\( R_H = \frac{2c}{\Omega_m^{0.4} H_0} \) (flat) |
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 = \frac{D_{LS}}{R_H} )</td>
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</tbody>
</table>
| \( \theta_2 \) \( \sim \) 4 \((\Omega)^{1/2}\) arcmin | Set by finite thickness of LSS  
\( \Delta z \sim 7 (\Omega h^2)^{-1/2} \) |
| \( \theta_3 \) \( \sim \) 9 \(h^{-1}\) arcmin | Set by horizon size at \( z_{eq} \), redshift of matter-radiation equality |

Peacock, “Cosmological Physics”, Ch 18, pg. 598
\[ \theta_1 \sim 1.8 \left( \Omega \right)^{1/2} \text{ degrees, which for an } \Omega=1 \text{ model, corresponds to 1.8 degrees and } l \sim 200 \text{ (since } l \sim 1/\theta) \]

At large angular scales \( \theta \gg \theta_1 \) (small \( l \)) the Sachs-Wolfe effect dominates the anisotropies. At small angular scales \( \theta < \theta_1 \) (large \( l \)), other contributions \( \sim 20\% \text{ amp. of } 1^{st} \text{ peak (doppler, damping, and finite thickness of the LSS) begin to dominate} \]

http://lambda.gsfc.nasa.gov/cgi-bin/cmbfast_form.pl
Two-Fluid Approximation

- This method (Seljak 1994), came before CMBFAST which is the exact generalization of this approximation method.

- **Approximations:**
  - Treat early universe as a single photon-baryon fluid
  - Ignore curvature, vector, tensor contributions
  - Simplified treatment of Thompson scattering
  - Valid only for adiabatic models (not isocurvature)

- **Method:**
  - Evolve two-fluid model until recombination epoch
  - Integrate over sources to obtain $C_l$

- Useful for an intuitive understanding of cosmological parameter dependencies ($C_l$ accurate to 10-20% over large range of $l$)
Two-Fluid Approximation cont...

\[ C_l = \langle |a_{lm}|^2 \rangle \quad \Delta(n) = \sum_{l,m} a_{lm} Y_{lm}(n) \]

\[ \Delta(n) = \int_0^{\tau_0} \left[ \mu \left( \phi + \frac{\delta_y}{4} + n \cdot v_b \right) + 2\dot{\phi} \right] e^{-\mu} d\tau \]

- Temperature
- Anisotropy

- Conformal Time
- Gravitational Potential
- Photon Density Perturbation
- Baryon Velocity
- Conformal Time Derivative
- Visibility Function
- Thompson Opacity along past light cone

- \( a = \text{scale factor} \)
- \( x_e = \text{ionization fraction} \)
- \( n_e = \text{electron # density} \)
- \( \sigma_T = \text{Thompson Cross Section} \)
- \( n \cdot v_b = -\partial \psi_b / \partial r \)
Physical Contributions to Source Term: Primary & Secondary Anisotropies

Two-Fluid Approx Source Term

Adiabatic Perturbations (Intrinsic Anisotropy)

Gravitational Perturbations (Sachs-Wolfe)

Velocity Perturbations (Doppler)

Integrated Sachs-Wolfe (ISW effect) (Important after recombination)

Where \( g = \mu e^{-\mu} \) is the Visibility Function

\[
S_T^{(S)}(k, \tau) = \left[ \dot{\mu} \left( \phi + \frac{\delta}{4} + n \cdot v_b \right) + 2\dot{\phi} \right] e^{-\mu}
\]
Physical Contributions to Source Term:
Exact CMBFAST Source Term

\[ S_T^{(S)}(k, \tau) = g\left( \Delta_{T0} + \psi + \frac{\dot{v}_b}{k} + \frac{\Pi}{4} + \frac{3\dot{\Pi}}{4k^2} \right) + e^{-\kappa(\phi + \psi)} + \dot{g}\left( \frac{v_b}{k} + \frac{3\Pi}{4k^2} \right) + \frac{3\dot{g}\Pi}{4k^2} \]

Adiabatic Perturbations (Intrinsic Anisotropy)
Gravitational Perturbations (Sachs-Wolfe)
Velocity Perturbations (Doppler)
Integrated Sachs-Wolfe (ISW effect)
(Important after recombination)

Photon Polarization Perturbations
(Anisotropic Thompson Scattering)

Where \( g(\tau) = \kappa \exp(-\kappa) \) is the Visibility Function
Two-Fluid Approximation cont...

Friedmann Equation

\[
(\ddot{a}/a)^2 = 8\pi G a^2 (\bar{\rho}_\gamma + \bar{\rho}_\nu + \bar{\rho}_b + \bar{\rho}_c)/3
\]

\[
y \equiv \frac{a}{a_{\text{eq}}} = (\alpha x)^2 + 2\alpha x, \quad x = \left(\frac{\Omega_m}{a_{\text{rec}}}\right)^{1/2} \frac{H_0 \tau}{2} \equiv \frac{\tau}{\tau_r}
\]

\[
y = \text{solution}, \text{ where } a = \text{scale factor, } x = \text{dimensionless time &}
\]

\[
a_{\text{eq}} = (\bar{\rho}_\gamma + \bar{\rho}_\nu)/(\bar{\rho}_b + \bar{\rho}_c) \approx 4.2 \times 10^{-5} \Omega_m^{-1} h^{-2}
\]

\[
a_{\text{rec}}^{-1} \approx 1100
\]

\[
\alpha^2 \equiv a_{\text{rec}}/a_{\text{eq}}
\]

\[
\Omega_m = \Omega_b + \Omega_c
\]

\[
h = H_0 / 100 \text{ km s}^{-1} \text{ Mpc}^{-1}
\]

Scale factor at recombination

Scale factor at matter-rad equality

Free parameter

Matter Density

Normalized Hubble Const. $H_0$
Tight Coupling Approx

• Approximate the LSS as an infinitely thin Delta function (i.e. $\mu >> 1$). (or a Gaussian in more realistic treatments)

• Good approx on scales larger than the Silk Damping scale.

• Treat photons-baryons as a single fluid with $v_b = v_\gamma$

• and perturbations given by: $\delta_b = \frac{3}{4} \delta_\gamma$

• Assume neutrino velocity and density perturbations scale with the photons. Invalid for small scales due to neutrino free streaming, but not significant for results.
Fourier Transformed Fluid Equations

\[ \delta_c = -\kappa v_c + 3\dot{\phi}, \quad \dot{v}_c = -\eta v_c + \kappa \phi \]
\[ \delta_y = -\frac{4}{3} \kappa v_y + 4\dot{\phi}, \quad \dot{v}_y = \frac{-\eta y_b v_y + \kappa \delta_y/3}{4/3 + y_b} + \kappa \phi \]
\[ \ddot{\phi} = -\eta \phi + \frac{3\eta^2[v_y(4/3 + y - y_c) + y_c v_c]}{2(1 + y)\kappa} \]

- Dimensionless time \(\chi = \frac{d}{dx}\)
- Dimensionless wave vector \(\kappa = \frac{k c \tau_r}{\zeta}\)
- Fourier Transformed Velocity Potential \(\psi\)
- Hubble parameter - essentially

\[ \eta \equiv \frac{\dot{a}}{a} = 2\alpha(\alpha x + 1)/(\alpha^2 x^2 + 2\alpha x) \]
\[ y_b \equiv \frac{\bar{\rho}_b}{\bar{\rho}_y} = 1.68 y \Omega_b/\Omega_m \]
\[ y_c = y \Omega_c/\Omega_m = (1 - \Omega_b/168\Omega_m)y \]

This is where we see the crucial \(\Omega_b\) and \(\Omega_m\) dependence from the cosmological parameters.
Two-Fluid Approximation cont...

**Initial Conditions**

\[
\phi = 1, \quad \delta_y = -2\phi \left(1 + \frac{3y}{16}\right), \quad \delta_e = \frac{3}{4} \delta_y
\]

\[
v_y = v_e = -\frac{\kappa}{\eta} \left[\frac{\delta_y}{4} + \frac{2\kappa^2(1 + y)\phi}{9\eta^2(4/3 + y)}\right]
\]

Satisfies: \(x<<1\) (universe is radiation dominated)
\(\kappa\eta <<1\) (mode is larger than Hubble radius)

Evolve Fluid Equations until

\[x_{\text{rec}} = \left[(\alpha^2 + 1)^{1/2} - 1\right]/\alpha\]

Apply Tight-Coupling Approx

\[\bar{\mu}e^{-\mu} \sim \text{delta function}\]
Two-Fluid Approximation cont...

Temperature Power Spectrum

\[ C_l = 4\pi A \int_0^\infty \kappa^n T(\kappa) D_l^2 \, d\ln \kappa \]
\[ D_l = \left( \phi + \frac{\delta_y}{4} + 2\Delta \phi \right) j_l(\kappa x_0) + v_\gamma j''_l(\kappa x_0) \]

Expanded in terms of
Multipole Moments \(D_l\)

- \(j_l\), \(j'_l\): Spherical Bessel function and its time derivative
- \(\kappa = kc\tau_r\): Dimensionless wave vector
- \(x_0\): Dimensionless angular distance to LSS
- \(P(k) = Ak^{-3}\kappa^{n-1}\): Primordial Power Spectrum (power law)
- \(T(\kappa) \propto \exp(-2\kappa^2 x_s^2)\): Silk Damping Transfer Function
- \(x_s = 0.6\Omega_m^{1/4}\Omega_b^{-1/2}a_{rec}^{3/4}h^{-1/2}\): Silk Damping Scale
- \(\Delta \phi = [2 - 8/y(x_{rec}) + 16x_{rec}/y^3(x_{rec})]/10y(x_{rec})\): ISW effect
Approximations to $C_l$ (thick) match us well (~10-20%) vs. exact calculations (thin), in the Two-Fluid Approximation
Two-Fluid Approximation cont…

Results – 6 important dimensionless parameters that uniquely determine the temp. power spectrum in this approximation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Suppression Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical Matter Density</td>
<td>$\Omega_m h^2$</td>
</tr>
<tr>
<td>Physical Baryon Density</td>
<td>$\Omega_b h^2$</td>
</tr>
<tr>
<td>Angular Distance to LSS</td>
<td>$x_o$</td>
</tr>
<tr>
<td>Slope of primordial power spectrum</td>
<td>$n$</td>
</tr>
<tr>
<td>Thompson opacity at recombination</td>
<td>$\mu(x_{\text{rec}})$</td>
</tr>
<tr>
<td>Silk Damping Scale</td>
<td>$x_s$</td>
</tr>
</tbody>
</table>

These 3 all suppress power on small scales relative to large scales

These are the 3 most interesting of the parameters

$\exp(-\mu(x_{\text{rec}}))$

$\exp(-2\kappa^2 x_s^2)$
Position of the First Acoustic Peak

Determined by angular size of acoustic horizon at decoupling, thus it depends mostly on $x_0$,

\[
x_0 = (\Omega_m a_{\text{rec}})^{-1/2} \quad \text{negligible cosmological constant}
\]

\[
x_0 = \Omega_m^{0.09} a_{\text{rec}}^{-1/2} \quad \text{negligible curvature}
\]

Position of first acoustic peak approximately $l \sim 6 x_0$
Seljak’s Plots

Fig. 2.—Anisotropy power spectra as a function of multipole moment $l$ for different cosmological models. In (a) curvature and vacuum energy dominated models with $\Omega_m = 0.25$ are compared to an $\Omega_m = 1$ model. In (b) $\Omega_m h^2$ is fixed at 0.5 and $\Omega_m h^2$ is varying, whereas in (c) $\Omega_m h^2$ is fixed at 0.05 and $\Omega_m h^2$ is varying. In (d) $\Omega_m$ is fixed and $h$ is varying. In (b), (c), and (d), $\Omega_m = 1$. In all cases varying the parameter changes the pattern of Doppler peaks.
### CMB Parameter Visualizations

<table>
<thead>
<tr>
<th>Uros Seljak’s Plots</th>
<th>Max Tegmark’s Movies</th>
<th>Wayne Hu’s Movies + 3D plots</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Baryons</strong></td>
<td><strong>Baryons</strong></td>
<td><strong>Baryons</strong></td>
</tr>
<tr>
<td><strong>Curvature, Lambda, and Matter</strong></td>
<td><strong>Curvature</strong></td>
<td><strong>Curvature and Lambda</strong></td>
</tr>
<tr>
<td></td>
<td><strong>Lambda</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Cold dark matter</strong></td>
<td><strong>Matter</strong></td>
</tr>
</tbody>
</table>

### I won’t cover these

<table>
<thead>
<tr>
<th>Hot dark matter (Neutrinos)</th>
<th>Tensor tilt</th>
<th>Reionization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar tilt</td>
<td>Tensor amplitude</td>
<td>Polarization (many movies)</td>
</tr>
<tr>
<td>Scalar amplitude</td>
<td>Bias</td>
<td></td>
</tr>
</tbody>
</table>

*Max Tegmark’s CMB Movies* [http://www.hep.upenn.edu/~max/index.html](http://www.hep.upenn.edu/~max/index.html)

*Wayne Hu’s CMB Movies* [http://background.uchicago.edu/~whu/physics/tour.html](http://background.uchicago.edu/~whu/physics/tour.html)

*Angelica de Oliveira-Costa’s Polarization Movies* [http://www.hep.upenn.edu/~angelica/polarization.html](http://www.hep.upenn.edu/~angelica/polarization.html)
Ω_m h^2 = 0.5 (fixed),
Just varying Ω_b
Increasing Ω_b
increases
power in odd numbered
peaks (1,3),
decreases
power in even peak 2.
Change in peak heights as the baryon content is raised (fixed $h=0.5$ and $\Omega_b h^2$ ranges from 0.00125 - 0.125) Baryons increase the amplitude of the oscillations as well as cause an alternation in the odd and even peak heights. (Hu & White (1996a))
Baryons $\Omega_b$ (Tegmark)

$\Omega_m$ held constant, baryon fraction $\Omega_b$ increased

The more baryons, the more wiggles, roughly speaking

Again, we see that it's actually mainly the odd-numbered peaks (1, 3, etc) that get boosted, which helps distinguish baryons from cold dark matter, which we'll discuss shortly

Indeed, you can see that the 2nd peak actually shrinks for a while. This is why the Boomerang and Maxima measurements favor lots of baryons since they don't see much of a 2nd peak.
Cold Dark Matter (CDM) (Tegmark)

- The CMB acoustic peaks result from a complicated interplay between dark matter and baryons.
- Increasing the CDM density pushes back the epoch of matter radiation equality to earlier in the age of the universe.
- All you really need to know here is that, as seen before, baryons boost the bumps, so increasing the CDM density reduces the baryon fraction and lowers the peaks.
Fixed $\Omega_b h^2$
Varying $\Omega_m h^2$
(Seljak 1994)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Omega_m$</th>
<th>$h^2 = \Omega_m h^2$</th>
<th>$\Omega_b h^2$</th>
<th>$\Omega_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RED – h=0.5</td>
<td>1</td>
<td>0.25</td>
<td>0.0125</td>
<td>0.05</td>
</tr>
<tr>
<td>GREEN – h=0.7</td>
<td>1</td>
<td>0.5</td>
<td>0.0125</td>
<td>0.025</td>
</tr>
<tr>
<td>BLUE – h=1</td>
<td>1</td>
<td>1</td>
<td>0.0125</td>
<td>0.0125</td>
</tr>
</tbody>
</table>
Degeneracies ($\Omega_m h^2$ and $\Omega_b h^2$)

- The dependence of Doppler peak positions and amplitudes on $\Omega_m h^2$ and $\Omega_b h^2$ yields a degeneracy, since both appear in the fluid evolution equations.

- The two parameters enter into the equations differently and have different physical effects:
  - $\Omega_b h^2$ – related to properties of photon-baryon plasma, determines the effective sound speed $c_s$ and horizon size at recombination
  - $\Omega_m h^2$ – related to the time evolution of the expansion factor, since it determines the epoch of matter radiation equality
Incr. $\Omega_b h^2$ and Decr. $\Omega_m h^2$ both lead to an increase in the 1st peak!

Increasing $\Omega_b h^2$ in (b) and decreasing $\Omega_m h^2$ in (c) increase the height of the 1st Doppler peak as shown: Increasing amplitudes are colored with Blue<Green< Red in both plots for clarity.

However, observations of the secondary peaks can break degeneracies...
Spatial curvature was completely irrelevant at \( z > 1000 \) (i.e. small angular scales and high \( l \)), when the acoustic oscillations were created.

\[
\Omega_k = 1 - \Omega_m - \Omega_\Lambda
\]  

\( \Omega_k \) therefore doesn't change the shape of the peaks at all - it merely shifts them sideways, since the conversion from the physical scale of the wiggles (in meters) into the angular scale (in degrees, or multipole \( l \)) depends on whether space is curved.

- If space has negative curvature (\( \Omega < 1, \Omega_k > 0 \)), like a Pringles potato chip or a saddle, then the angle subtended on the sky decreases, shifting the peaks to the right.
- If space has positive curvature, (\( \Omega > 1, \Omega_k < 0 \)), like a balloon, then the peaks shift to the left. (CLOSED)
The Cosmological Constant (Tegmark)

Just as with curvature, the cosmological constant was basically irrelevant in the early universe, when the acoustic oscillations were created. Matter and radiation still dominated over $\Lambda$.

It therefore doesn't change the shape of the peaks at all - it merely shifts them sideways, since it affects the conversion from the physical scale of the wiggles (in meters) into the angular scale (in degrees, or multipole $l$).

Note that this is analogous to the effect of increasing curvature, but goes in the opposite direction.

Just as with curvature, in addition to shifting the peaks sideways, a cosmological constant also causes fluctuations in the gravitational field to decrease over time, like spatial curvature in the previous movie. This is again known as the late ISW effect. Since it happens at late times, it shows up on large scales (to the left in the power spectrum).
Curvature, Lambda, & Matter (Seljak)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Omega_m$</th>
<th>$\Omega_\Lambda$</th>
<th>$\Omega_k = 1 - \Omega_m - \Omega_\Lambda$</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RED – Curvature Dominated</td>
<td>0.25</td>
<td>0</td>
<td>0.75</td>
<td>0.25</td>
</tr>
<tr>
<td>BLUE – Lambda Dominated</td>
<td>0.25</td>
<td>0.75</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>GREEN – Matter Dominated</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Angular diameter distance scaling with curvature and lambda
($\Omega_k = 1 - \Omega_o - \Omega_\Lambda$, fixed $\Omega_o$ h$^2$ and $\Omega_b$ h$^2$)
Examples from (Hu & White (1996a))
Preview of Christopher’s Talk
CMB Anisotropies Episode III:
Primordial Gravity Waves

- Polarization power spectra.

- Vector and tensor contributions to the temperature power spectrum (i.e. vorticity, & primordial gravity waves from inflation)

- How the E and B mode polarization spectra can break cosmological parameter degeneracies.
Conclusions

- The most interesting part of the CMB temperature power spectrum allows:
  - Simultaneous determination of $\Omega_b h^2$ and $\Omega_m h^2$
  - Constrains the $\Omega_m$ and $h$ parameter space

- In the curvature dominated model, the position of the first Doppler peak constrains $\Omega_m$

- In the vacuum energy dominated model, the position of the first Doppler peak does not constrain $\Omega_m$ but positions of the secondary peaks can be used to break the degeneracy.

- Ultimately, CMB power spectra measurements over a wide range of $l$ should tell us a great deal about $\Omega_m$, $\Omega_b$, and $h$

- Indeed WMAP has done it, but I’ll leave that to future talks!
References:

Papers:
Zaldarriaga, “An Introduction to CMB Anisotropies”, Lecture Notes, NYU
Ma & Bertschinger, 1995, ApJ., 455, 7M
Peacock, “Cosmological Physics” Ch 18

Useful CMB Websites:
Matias Zaldarriaga : http://physics.nyu.edu/matiasz/
Wayne Hu : http://background.uchicago.edu/~whu/index.html
Max Tegmark : http://www.hep.upenn.edu/~max/index.html
MAP: http://map.gsfc.nasa.gov/
LAMBDA: CMBFAST online interface http://lambda.gsfc.nasa.gov/cgi-bin/cmbfast_form.pl
Appendix

1. Review of the Equations (Boltzmann, Einstein, Fluid)

2. Line of Sight Method – How CMBFAST works
Review of the Equations

Temperature Anisotropy

Boltzmann Evolution
Equations
(scalar perturbations)

Hierarchy of Coupled
differential equations
(after expanding temp
anisotropy in multipole
moments)

\[
\Delta_T(k, n) = \sum_l (2l + 1)(-i)^l \Delta_{Tl} P_l(\mu)
\]

\[
\dot{\Delta}_T^{(s)} + i k \mu \Delta_T^{(s)} = \dot{\phi} - i k \mu \psi + \kappa [-\Delta_{T0}^{(s)} + \Delta_{T0}^{(s)} + i \mu \nu_b + \frac{1}{2} P_2(\mu) \Pi]
\]

\[
\dot{\Delta}_P^{(s)} + i k \mu \Delta_P^{(s)} = \kappa \{-\Delta_P^{(s)} + \frac{1}{2}[1 - P_2(\mu)] \Pi\}
\]

where \( \Pi = \Delta_{T2}^{(s)} + \Delta_{P2}^{(s)} + \Delta_P^{(s)} \)

\[
\Delta_T^{(s)} = -k \Delta_{T1}^{(s)} + \dot{\phi}
\]

\[
\Delta_T^{(s)} = \frac{k}{3} (\Delta_T^{(s)} - 2 \Delta_T^{(s)} + \psi) + \kappa \left( \frac{\nu_b}{3} - \Delta_T^{(s)} \right)
\]

\[
\Delta_T^{(s)} = \frac{k}{5} (2 \Delta_T^{(s)} - 3 \Delta_T^{(s)}) + \kappa \left( \frac{\Pi}{10} - \Delta_T^{(s)} \right)
\]

\[
\Delta_T^{(s)} = \frac{k}{2l + 1} \left[ l \Delta_T^{(s)} - (l + 1) \Delta_T^{(s)}(l+1) \right] - \kappa \Delta_T^{(s)}, \quad l > 2
\]

\[
\Delta_P^{(s)} = \frac{k}{2l + 1} \left[ l \Delta_P^{(s)} - (l + 1) \Delta_P^{(s)}(l+1) \right] + \kappa \left[ -\Delta_P^{(s)} + \frac{1}{2} \Pi \left( \delta_{l0} + \frac{\delta_{l2}}{5} \right) \right]
\]
Review of the Equations cont...

Fluid Equations

\[
\delta_c = -kv_c + 3\dot{\phi}, \quad \dot{v}_c = -\frac{\dot{a}}{a} v_c + k\psi
\]

\[
\delta_b = -kv_b + 3\dot{\phi},
\]

\[
\dot{v}_b = -\frac{\dot{a}}{a} v_b + c_s^2 k\delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} \kappa (3\Delta_{T1}^{(s)} - v_b) + k\psi
\]

Einstein Equations

\[
k^2 \phi + 3 \frac{\dot{a}}{a} \left( \dot{\phi} + \frac{\dot{a}}{a} \psi \right) = -4\pi G a^2 \delta \rho
\]

\[
k^2 \left( \phi + \frac{\dot{a}}{a} \psi \right) = 4\pi G a^2 \delta f
\]

Cold Dark Matter

Coupled Baryon-Photon Fluid

Poisson Equation to zero’th order

\(\delta \rho\) – matter source term

\(\delta f\) – momentum source term
Review of the Equations cont...

Temperature Anisotropy Power Spectrum

\[ C_{i}^{(s)} = (4\pi)^2 \int k^2 dk P_{\psi}(k) |\Delta_{Tl}^{(s)}(k, \tau = \tau_0)|^2 \]

Relation to Angular Correlation Function

\[ C(\theta) = \langle \Delta(n_1)\Delta(n_2) \rangle = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1) C_l P_l(\cos \theta) \quad n_1 \cdot n_2 = \cos \theta \]
Instead of solving the system of coupled differential equations, one may directly integrate the Boltzmann Evolution Equations along the photon past light cone to arrive at:

\[
\Delta_T^{(S)} = \int_0^{\tau_0} d\tau e^{i\mu(\tau - \tau_0)} e^{-\kappa} \times \left\{ \kappa \left[ \Delta_{T0} + i\mu \nu_b + \frac{1}{2} P_2 (\mu) \Pi \right] + \dot{\phi} - i\kappa \mu \psi \right\} \\
\Delta_P^{(S)} = -\frac{1}{2} \int_0^{\tau_0} d\tau e^{i\mu(\tau - \tau_0)} e^{-\kappa} \kappa [1 - P_2 (\mu)] \Pi
\]

\[
\Delta_{T,P}^{(S)} = \int_0^{\tau_0} d\tau e^{i\mu(\tau - \tau_0)} S_{T,P}^{(S)} (k, \tau)
\]

where \( \Pi = \Delta_{T2}^{(S)} + \Delta_{P2}^{(S)} + \Delta_{P0}^{(S)} \)

\[
S_T^{(S)} (k, \tau) = g \left( \Delta_{T0} + \psi + \frac{\dot{\nu}_b}{k} + \frac{\Pi}{4} + \frac{3\Pi}{4k^2} \right) + e^{-\kappa} (\dot{\phi} + \psi) + g \left( \frac{\nu_b}{k} + \frac{3\Pi}{4k^2} \right) + \frac{3\dot{\Pi}}{4k^2}
\]

\[
S_P^{(S)} (k, \tau) = \frac{3}{4k^2} \left[ g (k^2 \Pi + \dot{\Pi}) + 2\dot{g} \Pi + \dot{\Pi} \right]
\]
Physical Contributions to Source Term: Primary & Secondary Anisotropies

\[ S_T^{(s)}(k, \tau) = g\left( \Delta_{T0} + \psi + \frac{\dot{v}_b}{k} + \frac{\Pi}{4} + \frac{3\dot{\Pi}}{4k^2} \right) + e^{-\kappa(\phi + \psi)} + \dot{g}\left( \frac{v_b}{k} + \frac{3\dot{\Pi}}{4k^2} \right) + \frac{3\dot{g}\Pi}{4k^2} \]

Adiabatic Pertubations (Intrinsic Anisotropy)
Gravitational Perturbations (Sachs-Wolfe)
Velocity Perturbations (Doppler)
Integrated Sachs-Wolfe (ISW effect) (Important after recombination)
Photon Polarization Perturbations (Anisotropic Thompson Scattering)

Where \( g(\tau) = \dot{\kappa} \exp(-\kappa) \) is the Visibility Function

\[ \Pi = \Delta_{T2}^{(s)} + \Delta_{P2}^{(s)} + \Delta_{P0}^{(s)} \]
The Basis of CMBFAST

(Seljak & Zaldarriaga 1996)

This leads to the following familiar equation for $C_l$

$$C_l^{(S)} = (4\pi)^2 \int k^2 dk P_\psi(k) |\Delta_T^{(S)}(k, \tau = \tau_0)|^2$$

Where the multipole moments are now given by:

$$\Delta_{(T,P),l}^{(S)}(k, \tau = \tau_0) = \int_0^{\tau_0} S_{T,P}^{(S)}(k, \tau) j_l[k(\tau_0 - \tau)] d\tau$$

Source Term - which does not depend on the multipole $l$. – yields a relatively small system of diff. eq’s to evaluate

Geometrical Term – Spherical Bessel Function does not depend on the particular cosmological model – can be precomputed in advance